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Approximation of a Ball by Random Polytopes

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The mean volume and the mean surface area of the convex hull of n random points chosen independently and uniformly from the boundary of the d-dimensional unit ball are determined asymptotically as $n \to \infty$. The mean volume of the simplex with one vertex at the centre of a d-dimensional ball and with d random vertices chosen independently and uniformly from the boundary or from the interior of this ball is calculated.

1. INTRODUCTION

Let K be a d-dimensional convex body. For every integer $n > d$ there exists a polytope P_n which is best approximating for K with respect to the symmetric difference metric among all polytopes with at most n vertices. That means

$$\delta(K, P_n) = \min\{\delta(K, P) \mid P \text{ is a polytope with at most } n \text{ vertices}\},$$

where $\delta(K, P)$ denotes the d-dimensional volume of the set of all points belonging to one and only one of the bodies K and P.

Dudley [8] proved that for any convex body K,

$$\delta(K, P_n) \leqslant c_1(1/n)^{2/(d-1)} \qquad \text{for} \quad n > d.$$

On the other hand, if K is sufficiently smooth, a theorem of Gruber and Kenderov [13] shows that

$$\delta(K, P_n) \geqslant c_2(1/n)^{2/(d-1)} \qquad \text{for} \quad n > d.$$

Moreover, a conjecture of Gruber [11] says that the asymptotic relation

$$\delta(K, P_n) \sim c_3(1/n)^{2/(d-1)} \qquad \text{as} \quad n \to \infty$$

holds for sufficiently smooth K. Corresponding results are known for best approximating polytopes with respect to other measures of deviation (cf.

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Dudley [8], McClure and Vitale [16], Schneider [23]). In all these cases the rate of convergence is $(1/n)^{2/(d-1)}$. Gruber [11] gives a detailed survey of the approximation of convex bodies.

The best approximating polytopes P_n for a given convex body in general are not known explicitly. Therefore algorithms are needed to construct "well approximating" polytopes. Two deterministic procedures for constructing asymptotically $(n \to \infty)$ best approximating polygons for sufficiently smooth convex bodies in the plane were specified by McClure and Vitale [16]. A related problem was studied by Kenderov [15]. One method to get well approximating polytopes is to use random polytopes.

The convex hull $H_n(K)$ of n random points chosen independently and uniformly from the interior of a convex body K is a random polytope with at most n vertices approximating the body K. So it is interesting to investigate the mean values of some numbers associated with $H_n(K)$, such as the volume, the surface area, and the mean width. In particular the asymptotic behaviour $(n \to \infty)$ of these mean values is to be determined. Fundamental ideas are due to Rényi and Sulanke [20, 21]. In the last years important contributions to this subject have come from Schneider and Wieacker [24, 26] and Buchta [1, 2, 4]. Moreover, Buchta [5] gives a survey of approximation by random polytopes.

Let B_d denote the d-dimensional unit ball. The mean value of the symmetric difference $\delta(B_d, H_n(B_d))$ tends to zero as fast as $(1/n)^{2/(d+1)}$ if n tends to ∞. The expected difference between the surface areas of B_d and $H_n(B_d)$ tends to zero with the same order (Wieacker [26]; cf. [3, 19]). Both results hold if K is a sufficiently smooth body in the plane (Rényi and Sulanke [21]). If K is sufficiently smooth but not a ball, in higher dimensions it is only known that the expected deviation of K from $H_n(K)$ with respect to the mean width tends to zero as fast as $(1/n)^{2/(d+1)}$ if $n \to \infty$ (Schneider and Wieacker [24]). In all these cases the rate of convergence of best approximating polytopes is not attained.

Approximation of a convex body by random polytopes is improved if the random points are chosen independently and uniformly from the boundary of the convex body. If a sufficiently smooth d-dimensional convex body is approximated by random polytopes of this kind, the expected difference between the mean width of the convex body and the mean width of the random polytope tends to zero as fast as $(1/n)^{2/(d-1)}$ if $n \to \infty$ (Buchta *et al.* [7]). It is proved in the present paper that the same rate of convergence occurs in the case in which the d-dimensional unit ball is approximated by random polytopes of this type with respect to the symmetric difference metric. The corresponding result concerning the surface area deviation, which was cited in [7] without proof, is proved too, because it is needed as an auxiliary result to get the main result. That means that in all these cases the rate of convergence of best approximating polytopes is attained.

The above mentioned mean values of the surface area and of the mean width of random polytopes in the d-dimensional unit ball have been calculated for fixed n by Buchta *et al.* [6, 7]. The method used to derive these formulas does not work in the case of the volumes of random polytopes considered above (cf. Buchta [3, Chap. 2]). But it is easy to calculate the first and the second moment of a random simplex with one vertex at the centre of a d-dimensional ball and d random vertices chosen independently and uniformly from the boundary or from the interior of this ball. The mean volume of a random simplex of the last type characterises the ellipsoids among all centrally symmetric convex bodies. These results are stated in the last chapter of the present paper. Finally it should be remarked that most of the results of this paper are contained in the author's dissertation [19], with complete versions of the proofs.

2. The Surface Area of the Random Polytope

THEOREM 1. *Let S_n denote the surface area of the convex hull of n random points chosen independently and uniformly from the boundary ∂B_d of the d-dimensional unit ball B_d. The expected difference between S_n and the surface area ω_d of B_d is given by*

$$\bar{E}(\omega_d - S_n) = \frac{\omega_{d-1}}{2(d+1)(d-2)!}\left(\frac{\omega_{d-1}}{(d-1)\omega_d}\right)^{-(d+1)/(d-1)} \times \Gamma\left(d+\frac{2}{d-1}\right)\left(\frac{1}{n}\right)^{2/(d-1)}(1+o(1)) \qquad \text{as} \quad n \to \infty.$$

Proof. The surface area of the convex hull $\bar{H}_n$ of n random points is equal to the sum of the $(d-1)$-dimensional volumes of the facets of $\bar{H}_n$. The convex hull of d random points is a facet of $\bar{H}_n$ if and only if all remaining $n-d$ random points lie on only one of the two spherical caps determined by the d points. This happens with probability

$$\left(\frac{s}{\omega_d}\right)^{n-d} + \left(1 - \frac{s}{\omega_d}\right)^{n-d},$$

where s denotes the surface area of the smaller one of the two caps. The n random points are identically distributed and there are $\binom{n}{d}$ possibilities to choose d points out of n. So the expected surface area of $\bar{H}_n$ is given by

$$\bar{E}(S_n) = \binom{n}{d}\int_{x_1 \in \partial B_d} \cdots \int_{x_d \in \partial B_d}\left(\left(\frac{s}{\omega_d}\right)^{n-d} + \left(1-\frac{s}{\omega_d}\right)^{n-d}\right) T\frac{d\omega(x_1)}{\omega_d}\cdots\frac{d\omega(x_d)}{\omega_d},$$

where T denotes the $(d-1)$-dimensional volume of the convex hull of the points $x_1, ..., x_d$ and ω denotes the spherical surface measure of ∂B_d. The relation $s/\omega_d \leq \frac{1}{2}$ leads to

$$\bar{E}(S_n) = \binom{n}{d} \int_{x_1 \in \partial B_d} \cdots \int_{x_d \in \partial B_d} \left(1 - \frac{s}{\omega_d}\right)^{n-d} T \frac{d\omega(x_1)}{\omega_d} \cdots \frac{d\omega(x_d)}{\omega_d}$$
$$+ O\left(\left(\frac{1}{2}\right)^{n-d}\right) \quad \text{as} \quad n \to \infty.$$

There is an equivalent method to generate the random points $x_1, ..., x_d$ on ∂B_d: First, choose a random hyperplane with distance p from the origin and with unit normal vector $u \in \partial B_d$. Then, choose d random points $x'_1, ..., x'_d$ from the intersection of the random hyperplane with ∂B_d. The corresponding probability density functions for these procedures were specified by Miles [18] (cf. Buchta *et al.* [7]):

$$d\omega(x_1) \cdots d\omega(x_d) = (d-1)!\, T\, d\omega'(x'_1) \cdots d\omega'(x'_d)(1-p^2)^{-d/2}\, dp\, d\omega(u);$$

here ω' denotes the spherical surface measure on the intersection of the random hyperplane with ∂B_d. This transformation gives

$$\bar{E}(S_n) = \binom{n}{d} \frac{(d-1)!}{\omega_d^d} \int_{\partial B_d} \int_0^1 \left(1 - \frac{s}{\omega_d}\right)^{n-d} \left(\int \cdots \int T^2\, d\omega'(x'_1) \cdots d\omega'(x'_d)\right)$$
$$\times (1-p^2)^{-d/2}\, dp\, d\omega(u)(1+o(1)) \quad \text{as} \quad n \to \infty.$$

The second moment of the $(d-1)$-dimensional volume of a random simplex with vertices on the boundary of a $(d-1)$-dimensional ball of radius r is equal to $r^{2(d-1)}d/((d-1)!(d-1)^{d-1})$ (Miles [18]). So it is possible to evaluate the integral in brackets and to get

$$\bar{E}(S_n) = \binom{n}{d} d\omega_{d-1} \left(\frac{\omega_{d-1}}{(d-1)\omega_d}\right)^{d-1} \int_0^1 \left(1 - \frac{s}{\omega_d}\right)^{n-d}$$
$$\times (1-p^2)^{(d^2-d-2)/2}\, dp(1+o(1)) \quad \text{as} \quad n \to \infty.$$

Now a method of Wieacker [26] is used to determine the asymptotic behaviour of $\bar{E}(S_n)$:

The substitution $t = 1 - p$ introduces the height t of the smaller one of the two spherical caps determined by a hyperplane with distance p from the centre of the unit ball. It is sufficient to consider caps with small height $t \leq c$ (for suitable c). The value of the integral caused by caps with height $t > c$ exponentially tends to zero as $n \to \infty$. Hence,

$$\bar{E}(S_n) = \binom{n}{d} d\omega_{d-1} \left(\frac{\omega_{d-1}}{(d-1)\omega_d}\right)^{d-1} \int_0^c \left(1 - \frac{s(t)}{\omega_d}\right)^{n-d}$$
$$\times (2t - t^2)^{(d^2-d-2)/2}\, dt(1 + o(1)) \qquad \text{as} \quad n \to \infty.$$

Now the integrand in

$$I = \int_0^c \left(1 - \frac{s(t)}{\omega_d}\right)^{n-d} t^{(d^2-d-2)/2}(2-t)^{(d^2-d-2)/2}\, dt$$

has to be determined approximately for small values of t. The surface area $s(t)$ of a spherical cap with height t is given by

$$s(t) = \omega_{d-1} \int_0^t (2q - q^2)^{(d-3)/2}\, dq$$
$$= \omega_{d-1} \left(\frac{2}{d-1} 2^{(d-3)/2} t^{(d-1)/2} - \frac{d-3}{d+1} 2^{(d-5)/2} t^{(d+1)/2} + o(t^{(d+1)/2})\right)$$
$$\text{as} \quad t \to 0.$$

The binomial theorem leads to

$$\left(1 - \frac{s(t)}{\omega_d}\right)^{n'} = (1 - wt^{(d-1)/2})^{n'}$$
$$\times \left(1 + n' \frac{(d-3)(d-1)}{4(d+1)} wt^{(d+1)/2}(1 + o(1))\right) \qquad \text{as} \quad t \to 0$$

with

$$n' = n - d \qquad \text{and} \qquad w = 2^{(d-1)/2} \frac{\omega_{d-1}}{(d-1)\omega_d}.$$

After replacing $(2-t)^{(d^2-d-2)/2}$ by its linear part, the substitution $x = wn't^{(d-1)/2}$ yields

$$I = 2^{(d^2-d-4)/2} \int_0^{qn'} \left(1 - \frac{x}{n'}\right)^{n'} \left(1 + \frac{(d-3)(d-1)}{4(d+1)} \left(\frac{1}{wn'}\right)^{2/(d-1)}\right.$$
$$\left. \times x^{(d+1)/(d-1)}(1 + o(1))\right) \cdot \left(2 \left(\frac{1}{wn'}\right)^{(d^2-d-2)/(d-1)} x^{(d^2-d-2)/(d-1)}\right.$$
$$\left. - \frac{(d-2)(d+1)}{2} \left(\frac{1}{wn'}\right)^d x^d (1 + o(1))\right)$$
$$\cdot \frac{2}{d-1} \left(\frac{1}{wn'}\right)^{2/(d-1)} x^{(3-d)/(d-1)}\, dx$$

with $q = wc^{(d-1)/2}$. Now c should be small enough to obtain $0 < q \leqslant 1$. Multiplication of the factors in I gives a sum of integrals, each of which can be evaluated asymptotically using the asymptotic expansion (see Wieacker [26], Buchta [1])

$$\int_0^{qn} \left(1 - \frac{x}{n}\right)^n x^a \, dx = \Gamma(a+1) + O\left(\frac{1}{n}\right)$$

$$\text{as} \quad n \to \infty \qquad \text{for} \quad 0 < q \leqslant 1, \quad a > 0.$$

As a consequence of

$$n \sim n' \qquad \text{and} \qquad \binom{n}{d} \sim \frac{n^d}{d!}$$

the proof is finished. ∎

If the unit circle B_2 is approximated by the random polygon $\bar{H}_n$, according to Theorem 1, the mean perimeter deviation is given by

$$\bar{E}(\omega_2 - S_n) = 2\pi^3 \left(\frac{1}{n}\right)^2 (1 + o(1)) \qquad \text{as} \quad n \to \infty.$$

On the other hand, the regular n-gon with vertices on the boundary of B_2 is the best approximating inscribed n-gon for B_2 with respect to the perimeter deviation. Therefore the minimal perimeter deviation is given by

$$\omega_2 - 2n \sin\frac{\pi}{n} = \frac{\pi^3}{3}\left(\frac{1}{n}\right)^2 - \frac{\pi^5}{60}\left(\frac{1}{n}\right)^4 + \cdots.$$

That means that the expected perimeter deviation achieved by stochastical approximation is asymptotically ($n \to \infty$) 6 times the minimal perimeter deviation.

Remark 1 [26]. Let S_n denote the surface area of the convex hull of n random points chosen independently and uniformly from the interior of the d-dimensional unit ball B_d. The expected difference between S_n and the surface area ω_d of B_d is given by

$$E(\omega_d - S_n) = \frac{d\omega_{d-1}}{2(d+3)(d-1)!}\left(\frac{\omega_{d+1}}{(d+1)\omega_{d+2}}\right)^{-(d+3)/(d+1)}$$
$$\times \Gamma\left(d + \frac{2}{d+1}\right)\left(\frac{1}{n}\right)^{2/(d+1)} (1 + o(1)) \qquad \text{as} \quad n \to \infty.$$

3. The Volume of the Random Polytope

Theorem 2. *Let V_n denote the volume of the convex hull of n random points chosen independently and uniformly from the boundary ∂B_d of the d-dimensional unit ball B_d. The expected difference between V_n and the volume π_d of B_d is given by*

$$\bar{E}(\pi_d - V_n) = \frac{\omega_{d-1}}{2(d+1)!}\left(\frac{\omega_{d-1}}{(d-1)\omega_d}\right)^{-(d+1)/(d-1)} \times \Gamma\left(d+1+\frac{2}{d-1}\right)\left(\frac{1}{n}\right)^{2/(d-1)}(1+o(1)) \qquad as \quad n \to \infty,$$

where ω_d denotes the surface area of B_d.

Proof. The probability that the centre of B_d is not an interior point of the convex hull $\bar{H}_n$ of the n random points exponentially tends to zero as $n \to \infty$ (Wendel [25]). Therefore only those random polytopes which contain the centre of B_d in the interior have to be considered. In this case the volume V_n of $\bar{H}_n$ is given by

$$V_n = \sum \frac{1}{d} T(f)\, p(f)$$

(Hadwiger [14, p. 78]), where the sum runs over all facets f of $\bar{H}_n$, $T(f)$ denotes the $(d-1)$-dimensional volume of f, and $p(f)$ denotes the distance of the hyperplane containing f from the origin. Considerations similar to those at the beginning of the proof of Theorem 1 yield

$$\bar{E}(V_n) = \binom{n}{d} \int_{x_1 \in \partial B_d} \cdots \int_{x_d \in \partial B_d} \left(1 - \frac{s}{\omega_d}\right)^{n-d} \times \frac{1}{d} Tp \frac{d\omega(x_1)}{\omega_d} \cdots \frac{d\omega(x_d)}{\omega_d}(1+o(1)) \qquad \text{as} \quad n \to \infty,$$

where s, T, and p depend on the points $x_1, \ldots, x_d$ and ω denotes the spherical surface measure on B_d. Using the density transformation and the formula for the second moment of the volume of a random simplex with vertices on ∂B_d cited in the proof of Theorem 1, it follows that

$$\bar{E}(V_n) = \binom{n}{d} \omega_{d-1} \left(\frac{\omega_{d-1}}{(d-1)\omega_d}\right)^{d-1} \int_0^1 \left(1 - \frac{s}{\omega_d}\right)^{n-d} \times p(1-p^2)^{(d^2-d-2)/2}\, dp(1+o(1)) \qquad \text{as} \quad n \to \infty.$$

The substitution $t = 1 - p$ leads to

$$\bar{E}(V_n) = \binom{n}{d} \omega_{d-1} \left(\frac{\omega_{d-1}}{(d-1)\omega_d} \right)^{d-1} \int_0^1 \left(1 - \frac{s(t)}{\omega_d} \right)^{n-d}$$

$$\times (1-t)(2t-t^2)^{(d^2-d-2)/2} \, dt (1+o(1))$$

$$= \frac{1}{d} \bar{E}(S_n)(1+o(1)) - \binom{n}{d} \omega_{d-1} \left(\frac{\omega_{d-1}}{(d-1)\omega_d} \right)^{d-1}$$

$$\times \int_0^1 \left(1 - \frac{s(t)}{\omega_d} \right)^{n-d} t(2t-t^2)^{(d^2-d-2)/2} \, dt (1+o(1)) \qquad \text{as} \quad n \to \infty.$$

The asymptotic behaviour of the mean surface area $\bar{E}(S_n)$ of $\bar{H}_n$ determined in Theorem 1 appears in the first term. The integral in the second term is calculated in the same way as the corresponding integral in the proof of Theorem 1: It is sufficient to consider spherical caps with small height $t \leqslant c$ for suitable c. The surface area $s(t)$ of a spherical cap with height t is approximated by the first term of the corresponding polynomial in the proof of Theorem 1. After using the substitution mentioned in that proof the application of the asymptotic expansion of the gamma function finishes the proof. ∎

In the plane and in three dimensions it is possible to compare the approximation of B_d by random polytopes $\bar{H}_n$ with optimal approximation of B_d by inscribed polytopes having at most n vertices. The regular n-gon with vertices on the boundary of the unit ball B_2 has maximal area among all n-gons contained in B_2. So the minimal distance between B_2 and inscribed n-gons with respect to the symmetric difference is given by

$$\pi - \frac{n}{2} \sin \frac{2\pi}{n} = \frac{2}{3} \pi^3 \left(\frac{1}{n} \right)^2 - \frac{2}{15} \pi^5 \left(\frac{1}{n} \right)^4 + \cdots.$$

Theorem 2 shows that

$$\bar{E}(\pi - V_n) = 4\pi^3 \left(\frac{1}{n} \right)^2 \left(1 + o(1) \right) \qquad \text{as} \quad n \to \infty.$$

Therefore the expected distance achieved by stochastic approximation is asymptotically 6 times the minimal distance.

If the three-dimensional unit ball B_3 is approximated by inscribed polytopes with at most n vertices the minimal difference of the volumes is given by

$$\frac{4}{\sqrt{3}} \pi^2 \frac{1}{n} (1 + o(1)) \qquad \text{as} \quad n \to \infty$$

(Fejes Tóth [9], Gruber [12]). On the other hand, using approximation by random polytopes the mean difference is (Theorem 2)

$$\bar{E}(\pi_3 - V_n) = 16\pi \frac{1}{n}(1 + o(1)) \qquad \text{as} \quad n \to \infty.$$

So the expected distance obtained by stochastic approximation in three dimensions is asymptotically $4\sqrt{3}/\pi$ (≈ 2.21) times the minimal distance.

Remark 2 [26, 19]. Let V_n denote the volume of the convex hull of n random points chosen independently and uniformly from the interior of the d-dimensional unit ball B_d. The expected difference between V_n and the volume π_d of B_d is given by

$$E(\pi_d - V_n) = \frac{\omega_{d-1}}{2(d+3)(d-1)(d-1)!}\left(\frac{\omega_{d+1}}{(d+1)\omega_{d+2}}\right)^{-(d+3)/(d+1)} \times \Gamma\left(d+1+\frac{2}{d+1}\right)\left(\frac{1}{n}\right)^{2/(d+1)}(1+o(1)) \qquad \text{as} \quad n \to \infty,$$

where ω_d denotes the surface area of B_d.

4. Characterization of Ellipsoids by Random Simplices

THEOREM 3. (a) *Let Z denote the volume of a random simplex with one vertex at the centre of the d-dimensional unit ball B_d and d vertices chosen independently and uniformly from the boundary of B_d. The first and the second moments of Z are given by*

$$\bar{E}(Z) = \frac{\omega_d}{d}\left(\frac{\omega_{d-1}}{(d-1)\omega_d}\right)^d \qquad \textit{and} \qquad \bar{E}(Z^2) = \frac{1}{d!\, d^d},$$

where ω_d denotes the surface area of B_d.

(b) *Let Z denote the volume of a random simplex with one vertex at the centre of B_d and d vertices chosen independently and uniformly from the interior of B_d. The first and the second moments of Z are given by*

$$E(Z) = \pi_d\left(\frac{\pi_{d-1}}{(d+1)\pi_d}\right)^d \qquad \textit{and} \qquad E(Z^2) = \frac{1}{d!(d+2)^d},$$

where π_d denotes the volume of B_d.

Proof. (a) The volume Z is represented by $Z = pT/d$, where T denotes the $(d-1)$-dimensional volume of the $(d-1)$-dimensional simplex with random vertices $x_1, ..., x_d \in \partial B_d$ and p denotes the distance from the centre of B_d to the hyperplane determined by $x_1, ..., x_d$. So

$$\bar{E}(Z) = \frac{1}{d} \int_{x_1 \in \partial B_d} \cdots \int_{x_d \in \partial B_d} pT \frac{d\omega(x_1)}{\omega_d} \cdots \frac{d\omega(x_d)}{\omega_d}$$

$$= \frac{(d-1)!}{d\omega_d^d} \int_{\partial B_d} \int_0^1 \left(\int \cdots \int T^2 \, d\omega'(x'_1) \cdots d\omega'(x'_d) \right)$$

$$\times p(1-p^2)^{-d/2} \, dp \, d\omega(u),$$

where the coordinate transformation and the notations of the proof of Theorem 1 are used. The d-fold integral in brackets is evaluated using the second moment of the $(d-1)$-dimensional volume of a random simplex with all vertices on the boundary of a $(d-1)$-dimensional ball (cf. proof of Theorem 1). Finally the substitution $q = 1 - p^2$ is used to finish the proof. The second moment $\bar{E}(Z^2)$ is calculated in a similar way. Now the third moment of the $(d-1)$-dimensional volume of a random simplex with all vertices on the boundary of a $(d-1)$-dimensional ball is used (see Miles [18]). The substitution $p = \cos x$ and integration by parts finish the proof.

(b) The well-known coordinate transformation (see Santaló [22])

$$dx_1 \cdots dx_d = (d-1)! \, T \, dx'_1 \cdots dx'_d \, dp \, d\omega(u)$$

yields

$$E(Z) = \frac{1}{d} \int_{x_1 \in B_d} \cdots \int_{x_d \in B_d} pT \frac{dx_1}{\pi_d} \cdots \frac{dx_d}{\pi_d}$$

$$= \frac{(d-1)!}{d\pi_d^d} \int_{\partial B_d} \int_0^1 \left(\int \cdots \int T^2 \, dx'_1 \cdots dx'_d \right) p \, dp \, d\omega(u).$$

Here dx' denotes the volume element of the intersection of B_d with a hyperplane determined by the normal vector $u \in \partial B_d$ and distance p from the origin. The integral in brackets is evaluated using the second moment $r^{2(d-1)}d/((d-1)!(d+1)^{d-1})$ (Miles [18]) of the $(d-1)$-dimensional volume of a simplex with all d vertices chosen independently and uniformly from the interior of a $(d-1)$-dimensional ball of radius r. The third moment of the volume of such a random simplex (Miles [18]) is used for $E(Z^2)$. The integration methods mentioned above (a) finish the proof. ∎

Remark 3. A special version of a theorem proved by Groemer [10] yields the following result: Let K denote a d-dimensional centrally symmetric convex body with centre 0 and with fixed volume π_d. Let Z denote the volume of a random simplex with one vertex at 0 and d vertices chosen independently and uniformly from the interior of K. For any real number $m \geqslant 1$, the mth moment

$$E_K(Z^m) = \int_{x_1 \in K} \cdots \int_{x_d \in K} Z^m \frac{dx_1}{\pi_d} \cdots \frac{dx_d}{\pi_d}$$

of Z is minimal among all d-dimensional centrally symmetric convex bodies of volume π_d if and only if K is an ellipsoid.

COROLLARY. *For every d-dimensional centrally symmetric convex body K of fixed volume* π_d *the inequalities*

$$E_K(Z) \geqslant \pi_d \left(\frac{\pi_{d-1}}{(d+1)\pi_d} \right)^d \quad \text{and} \quad E_K(Z^2) \geqslant \frac{1}{d!(d+2)^d}$$

hold with equality if and only if K is an ellipsoid.

Remark 4. McKinney [17] proved a more general version of the following result: Let Z denote the volume of a simplex with one vertex at the centre and d vertices on the boundary ∂K of a d-dimensional centrally symmetric convex body K. Then for every K of volume π_d the inequality

$$\operatorname{Max}\{Z \mid x_1, ..., x_d \in \partial K\} \geqslant 1/d!$$

holds with equality if and only if K is an ellipsoid.

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